Moduli space of Fedosov structures

Stanislav Dubrovskiy

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1 Introduction

Fedosov space is a triple: a manifold M^{2n} with a symplectic structure ω, and a compatible symmetric connection Γ: (M, ω, Γ). Compatibility means that ω is preserved under geodesic flow of Γ:

$$\nabla \omega = 0. \tag{1.1}$$

There is a canonical quantization for these manifolds, see [GRS] and references therein.

Here we are interested in local invariants of a Fedosov structure. Namely, we take a space $\mathcal{F}$, of germs of Fedosov structures at a point, and act on them by local coordinate changes, that is the group of all diffeomorphisms

$$G := \mathrm{Diff}(\mathbb{R}^{2n}, 0)$$

fixing the point. A quotient of $\mathcal{F}$ by this action is called the moduli space of Fedosov structures:

$$\mathcal{M} = \mathcal{F}/\mathrm{Diff}(\mathbb{R}^{2n}, 0).$$

This action can be restricted from space of germs $\mathcal{F}$ to space of k-jets of Fedosov structures, $\mathcal{F}_k$, with corresponding quotient:

$$\mathcal{M}_k = \mathcal{F}_k/\mathrm{Diff}(\mathbb{R}^{2n}, 0)$$

called the moduli space of k-jets. We will only work with generic Fedosov structures. For a generic orbit $\mathcal{O}_k$, its dimension:

$$\dim \mathcal{O}_k = \mathrm{codim}\ G_\Phi$$

is the codimension of the stabilizer G_Φ of a generic Fedosov structure Φ in G. Then we will call

$$\dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k,$$

and construct the *Poincaré series* of $\mathcal{M}$:

$$p_\Phi(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty} (\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1}) t^k$$

Theorem 1.1 *Poncaré series coefficients are polynomial in k, and the series has the form:*

$$p_\Phi(t) = \frac{n[8n(2n^2-1)(n+1)+11]}{6} + \frac{n(2n+1)[4n^4+2n^3-6n^2-4n-3]}{3}t +$$
$$+(t-t^2)\delta^2_{2n} + 2n\sum_{k=2}^{\infty}\left[2\binom{2n+2}{4}\binom{2n+k-1}{2n-1} - \binom{2n+k+1}{2n-1}\right]t^k.$$

It represents a rational function.

Remark This confirms the assertion of Tresse, cf. [T], that algebras of "natural" differential-geometric structures are finitely-generated.

Proof Postponed until section 5.

Similar results for other differential-geometric structures were obtained earlier in [Sh] and [D].

To explain significance of Poincaré series represented by a rational function, we make the following:

Remark If a geometric structure is described by a finite number of functional moduli, then its Poncaré series is rational. In particular, if there are m functional invariants in n variables, then

$$p(t) = \frac{m}{(1-t)^n}$$

Indeed, dimension of moduli spaces of k-jets is just the number of monomials up to the order k in the formal power series of the m given invariants:

$$\dim \mathcal{M}_k = m\binom{n+k}{n}$$

For more details and slightly more general formulation see **Theorem 2.1** in [Sh2].

2 Action formulas

As usual, two C^∞-functions on $\mathbb{R}^{2n}$ have the same k-jet at a point if their first k derivatives are equal in any local coordinates.

We say that two connections ∇ and $\tilde{\nabla}$ have the same k-jet at 0 if for any two C^∞-vector fields X, Y and any C^∞-function f, the functions $\nabla_X Y(f)$ and $\tilde{\nabla}_X Y(f)$ have the same k-jet at 0. This is equivalent to connection coefficients of ∇ and $\tilde{\nabla}$ having the same k-jet. We denote by $j^k\Gamma$ the k-jet of Γ.

There is an action of the group of germs of origin-preserving diffeomorphisms

$$G = \mathrm{Diff}(\mathbb{R}^{2n}, 0)$$

on $\mathcal{F}$ and $\mathcal{F}_k$.

For $\varphi \in G$, (ω, ∇)(or (ω, Γ)) $\in \mathcal{F}$ and $j^k\Gamma \in \mathcal{F}_k$:

$$\Gamma \mapsto \varphi^*\Gamma\,, \quad j^k\Gamma \mapsto j^k(\varphi^*\Gamma)\,,$$

where

$$(\varphi^*\nabla)_X Y = \varphi_*^{-1}(\nabla_{\varphi_* X}\, \varphi_* Y)$$

Let us introduce a filtration of G by normal subgroups:

$$G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \dots,$$

where

$$G_k = \{\, \varphi \in G \mid \varphi(x) = x + (\varphi_1(x), \dots \varphi_n(x)),\ \varphi_i = O(|x|^k),\ i = 1, \dots, 2n\,\}$$

The subgroup G_k acts trivially on $\mathcal{F}_p$ for $k \geq p+3$.
It means that the action of G coincides with that of G/G_{p+3} on each $\mathcal{F}_p$.

Now G/G_{p+3} is a finite-dimensional Lie group, which we will call K_p .
Denote by $\mathrm{Vect}_0(\mathbb{R}^{2n})$ the Lie algebra of C^∞-vector fields, vanishing at the origin. It acts on $\mathcal{F}$ as follows:

Definition 2.1 *For $V \in \mathrm{Vect}_0(\mathbb{R}^{2n})$ generating a local 1-parameter subgroup g^t of* $\mathrm{Diff}(\mathbb{R}^n, 0)$, the Lie derivative *of a connection ∇ in the direction V is a (1,2)-tensor:*

$$\mathcal{L}_V \nabla = \frac{d}{dt}\Big|_{t=0} {g^t}^* \nabla$$

Lemma 2.2

$$(\mathcal{L}_V \nabla)(X, Y) = [V, \nabla_X Y] - \nabla_{[V,X]} Y - \nabla_X [V, Y] \tag{2.2}$$

Proof Below the composition $\circ$ is understood as that of differential operators acting on functions.

$$(\mathcal{L}_V \nabla)(X, Y) = \frac{d}{dt}\Big|_{t=0} g_*^{-t}[\nabla_{g_*^t X}\, g_*^t Y] = \frac{d}{dt}\Big|_{t=0} \left[(g^t)^* \circ [\nabla_{g_*^t X}\, g_*^t Y] \circ (g^{-t})^* \right] =$$

$$\frac{d}{dt}\Big|_{t=0} (g^t)^* \circ \nabla_X Y + \nabla_X Y \circ \frac{d}{dt}\Big|_{t=0} (g^{-t})^* + \nabla_{\frac{d}{dt}|_{t=0} g_*^t X} Y + \nabla_X \frac{d}{dt}\Big|_{t=0} g_*^t Y =$$

$$V \circ \nabla_X Y - \nabla_X Y \circ V - \nabla_{\frac{d}{dt}|_{t=0} g_*^{-t} X} Y - \nabla_X \frac{d}{dt}\Big|_{t=0} g_*^{-t} Y =$$

$$\mathcal{L}_V(\nabla_X Y) - \nabla_{\mathcal{L}_V X} Y - \nabla_X(\mathcal{L}_V Y)$$

□

This defines the action on the germs of connections. Now we can define the action of $\mathrm{Vect}_0(\mathbb{R}^{2n})$ on jets $\mathcal{F}_k$. For $V \in \mathrm{Vect}_0(\mathbb{R}^{2n})$:

$$\mathcal{L}_V(j^k\Gamma) = j^k(\mathcal{L}_V \Gamma)\ ,$$

where Γ on the right is an arbitrary representative of the $j^k\Gamma$ on the left.
This is well-defined, since in the coordinate version of (2.2):

$$(\mathcal{L}_V\Gamma)^l_{ij} = V^k\frac{\partial\Gamma^l_{ij}}{\partial x^k} - \Gamma^k_{ij}\frac{\partial V^l}{\partial x^k} + \Gamma^l_{kj}\frac{\partial V^k}{\partial x^i} + \Gamma^l_{ik}\frac{\partial V^k}{\partial x^j} + \frac{\partial^2 V^l}{\partial x^i\partial x^j} \tag{2.3}$$

elements of k-th order and less are only coming from $j^k\Gamma$, because $V(0) = 0$.
Einstein summation convention in (2.3) above and further on is assumed.
Consequently, the action is invariantly defined. This can also be expressed as commutativity of the following diagram:

$$\begin{array}{ccccccccc} j^0\mathcal{F} & \longleftarrow & \dots & \longleftarrow & j^{k-1}\mathcal{F} & \xleftarrow{\pi_k} & j^k\mathcal{F} & \longleftarrow & \dots & \longleftarrow & \mathcal{F} \\ \downarrow\mathcal{L}_V & & & & \downarrow\mathcal{L}_V & & \downarrow\mathcal{L}_V & & & & \downarrow\mathcal{L}_V\ , \\ j^0\Pi & \longleftarrow & \dots & \longleftarrow & j^{k-1}\Pi & \xleftarrow{\pi_k} & j^k\Pi & \longleftarrow & \dots & \longleftarrow & \Pi \end{array}$$

where π_k is projection from k-jets onto $(k-1)$-jets, $\mathcal{F}$ and Π denote spaces of germs of connections and that of (1,2)-tensors respectively, at 0.

3 Stabilizer of a generic k-jet

[The following discussion closely mirrors that of section 3 in [D].]
Dimensions of stabilizers of generic k-jets G_Φ are required to find orbit dimensions for orbits $\mathcal{O}_k$ of generic k-jets. The subalgebra generating G_Φ consists of those $V \in \mathrm{Vect}_0(\mathbb{R}^{2n})$ that

$$\mathcal{L}_V(j^k\Phi) = 0.$$

Since $\Phi = (\omega, \Gamma)$, this entails two conditions:

$$\mathcal{L}_V(j^k\omega) = 0 \qquad\qquad \mathcal{L}_V(j^k\Gamma) = 0. \tag{3.4}$$

In the next two sections devoted to finding the stabilizer G_Φ we assume that ω is reduced to canonical symplectic form in Darboux coordinates. In these coordinates compatibility (1.1) is written as:

$$\omega_{i\alpha}\Gamma^\alpha_{kj} = \omega_{j\beta}\Gamma^\beta_{ki},. \tag{3.5}$$

where $\omega = J = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix}$, a standard symplectic matrix, cf. [GRS], p.110.
We can introduce grading in homogeneous components on V:

$$V = V_1 + V_2 + \dots$$

$$(\ V_0 = 0, \text{ so that } V \text{ preserve the origin }) \ ,$$

on Γ:

$$\Gamma = \Gamma_0 + \Gamma_1 + \dots\ ,$$

and on ω:

$$\omega = \omega_0\ ,$$

where $\omega_0 = J$ is a standard symplectic form.
Then (3.4) is rewritten as follows:

$$\mathcal{L}_V(j^k\omega) = \mathcal{L}_{V_1+V_2+\ldots}(\omega_0) = 0$$

$$\mathcal{L}_V(j^k\Gamma) = j^k\mathcal{L}_V(\Gamma) = j^k\mathcal{L}_{V_1+V_2+\ldots}(\Gamma_0+\Gamma_1+\ldots+\Gamma_k+\ldots) =$$

$$= \underbrace{\mathcal{L}_{V_1}\Gamma_0 + \frac{\partial^2 V_2}{\partial x^2}}_{\text{0th order}} + \underbrace{\mathcal{L}_{V_1}\Gamma_1 + \tilde{\mathcal{L}}_{V_2}\Gamma_0 + \frac{\partial^2 V_3}{\partial x^2}}_{\text{1st order}} + \ldots$$

$$\ldots + \underbrace{\tilde{\mathcal{L}}_{V_{k+1}}\Gamma_0 + \tilde{\mathcal{L}}_{V_k}\Gamma_1 + \ldots + \mathcal{L}_{V_1}\Gamma_k + \frac{\partial^2 V_{k+2}}{\partial x^2}}_{\text{k-th order}} \quad ,$$

$$\text{where} \quad \left(\frac{\partial^2 V_2}{\partial x^2}\right)^l_{ij} = \frac{\partial^2 V_2^l}{\partial x^i x^j}$$

and

$$\tilde{\mathcal{L}}_V\Gamma = \mathcal{L}_V\Gamma - \frac{\partial^2 V}{\partial x^2} \ .$$

$\tilde{\mathcal{L}}_V\Gamma$ with indexes looks like this:

$$(\tilde{\mathcal{L}}_V\Gamma)^l_{ij} = V^k\frac{\partial\Gamma^l_{ij}}{\partial x^k} - \Gamma^k_{ij}\frac{\partial V^l}{\partial x^k} + \Gamma^l_{kj}\frac{\partial V^k}{\partial x^i} + \Gamma^l_{ik}\frac{\partial V^k}{\partial x^j} \ ,$$

so $\tilde{\mathcal{L}}_V\Gamma$ is just the first 4 terms of $(\mathcal{L}_V\Gamma)$, cf.(2.3).
The stabilizer condition therefore results in a system:

$$\begin{cases} \mathcal{L}_{V_1}\omega_0 = 0 & \\ \mathcal{L}_{V_2}\omega_0 = 0 & \\ \quad\vdots & \omega - \text{part} \\ \mathcal{L}_{V_{k+1}}\omega_0 = 0 & \\ ----------------------- & \\ \mathcal{L}_{V_1}\Gamma_0 + \dfrac{\partial^2 V_2}{\partial x^2} = 0 & \\ \mathcal{L}_{V_1}\Gamma_1 + \tilde{\mathcal{L}}_{V_2}\Gamma_0 + \dfrac{\partial^2 V_3}{\partial x^2} = 0 & \Gamma - \text{part} \\ \quad\vdots & \\ \mathcal{L}_{V_1}\Gamma_k + \tilde{\mathcal{L}}_{V_2}\Gamma_{k-1} + \ldots + \tilde{\mathcal{L}}_{V_{k+1}}\Gamma_0 + \dfrac{\partial^2 V_{k+2}}{\partial x^2} = 0 & \end{cases} \tag{3.6}$$

Our present goal is finding all $(V_1, V_2, \ldots, V_{k+2})$ solving the above system for a generic Φ. Let us start with the Γ-part. Assuming V_1 is arbitrary, we can uniquely find V_2 from the first equation, as guaranteed by the following lemma on mixed derivatives:

Lemma 3.1 *Given a family* $\{f_{ij}\}_{1\le i,j\le N}$ *of smooth functions, solution* u *for the system:*

$$\begin{cases} u_{,kl} = f_{kl} \\ 1 \le k,l \le N \end{cases}$$

(indexes after a comma henceforth will denote differentiations in corresponding variables) exists if and only if

$$\begin{cases} f_{ij} = f_{ji} \\ f_{ij,k} = f_{kj,i} \end{cases} \tag{3.7}$$

If f_{ij} *are homogeneous polynomials of degree* $s \ge 0$*, then* u *can be uniquely chosen as a polynomial of degree* $s+2$*.*

Proof is a straightforward integration of the right-hand sides. □

Therefore, if we treat highest-order V_k in each equation in Γ-part of (3.6) as an unknown, we see that various (combinations of) $\mathcal{L}_V\Gamma$ must satisfy (3.7). The first condition is satisfied automatically since Γ is symmetric. The second one gives:

$$(\mathcal{L}_V\Gamma)^l_{ij,p} = (\mathcal{L}_V\Gamma)^l_{pj,i}$$

This condition for the first equation in Γ-part of (3.6) is satisfied trivially, since V_1 is of the first degree in x, and Γ_0 is constant. Hence, V_2 exists and, since it must be of the second degree, is unique. However, if $k \ge 1$ (so there is need for more than one equation) there is a non-trivial condition on the second equation:

$$\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0 + \frac{\partial^2 V_3}{\partial x^2} = 0$$

It follows from the Lemma 3.1 that for the existence of V_3 it is necessary (and sufficient) to have the following condition:

$$(\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0)_{ij,p} = (\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0)_{pj,i}\ ,\ i < p \tag{3.8}$$

Outside exceptional dimension two this condition fails for a generic connection unless $V_1 = 0$. In other words (3.8), considered as a condition on V_1 implies $V_1 = 0$ (and hence $V_2 = V_3 = ... = 0$). The rest is a proof of this assertion.

Let us consider (3.8) as a linear homogeneous system on the components of V_1. We will present a Fedosov structure for which (3.8) is non-degenerate. Since nondegeneracy is an open condition on the space $\mathcal{F}_k$, the same (namely non-degeneracy and resulting trivial solution $V = 0$ for the stabilizer) would hold for a generic structure.

Since symplectic part of the structure is fixed by our choice to work in symplectic coordinates, we need only to present the corresponding connection 1-jet. In this 1-jet we choose to have $\Gamma_0 = 0$, which simplifies (3.8) to:

$$(\mathcal{L}_{V_1}\Gamma_1)^l_{ij,p} = (\mathcal{L}_{V_1}\Gamma_1)^l_{pj,i}$$

Let us expand it using (2.3):

$$(\Gamma^l_{1ij,k} - \Gamma^l_{1kj,i})V^k_{1,p} + (\Gamma^l_{1kj,p} - \Gamma^l_{1pj,k})V^k_{1,i} +$$
$$(\Gamma^k_{1pj,i} - \Gamma^k_{1ij,p})V^l_{1,k} + (\Gamma^l_{1ik,p} - \Gamma^l_{1pk,i})V^k_{1,j} = 0 \ , \ i < p \qquad (3.9)$$

Recall that summation over repeated indexes above is assumed.

In local symplectic coordinates:

$$V^k_1 = \sum_{s=1}^{n} b^k_s x^s \ , \qquad \Gamma^l_{1ij} = \sum_{m=1}^{n} c^{lm}_{ij} x^m, \quad c^{lm}_{ij} = c^{lm}_{ji} \text{ (connection is symmetric)},$$

and (3.9) becomes the system on b^k_s:

$$(c^{lk}_{ij} - c^{li}_{kj})b^k_p + (c^{lp}_{kj} - c^{lk}_{pj})b^k_i + (c^{ki}_{pj} - c^{kp}_{ij})b^l_k + (c^{lp}_{ik} - c^{li}_{pk})b^k_j = 0 \ , \ i < p \quad (3.10)$$

The requirement (1.1) on Γ to be a symplectic connection is passed through to each of its homogeneous components $\Gamma_{\mathbf{k}}$ as the following symmetry condition:

$$\omega_{i\alpha}\Gamma^\alpha_{\mathbf{k}jl} = \omega_{l\alpha}\Gamma^\alpha_{\mathbf{k}ji} \ ,$$

that can be thought of as 'Γ with lowered indexes' is completely symmetric (cf. the discussion on p.110 (especially equation (1.5)) in [GRS]).
Of course $\omega_{i\alpha}$ in our setting is just the standard symplectic matrix. Another way to think about it in terms of e.g. coefficients of Γ_1 is that they 'form a symplectic matrix', namely $\Gamma_1 \in \mathtt{sp}(2n)$ in the left upper and lower indexes:

$$\begin{array}{rcl} c^{im}_{Jk} & = & c^{\bar{J}m}_{\bar{i}k} \\ c^{Im}_{jk} & = & c^{\bar{j}m}_{\bar{I}k} \\ c^{Im}_{Jk} & = & -c^{\bar{J}m}_{\bar{I}k} \ \Rightarrow \ c^{im}_{jk} \ = \ -c^{\bar{j}m}_{\bar{i}k} \end{array} \qquad (3.11)$$

$$\forall m, k \in [1, \dots 2n], \ i, j \in [1 \dots n], \ I, J \in [n+1, \dots 2n], \ \bar{i} = i + n, \bar{I} = I - n$$

We will also consider only such Γ_1 that

$$c^{lp}_{ij} \neq 0 \text{ only if } \{i, j, l, p\} = \{\alpha, \beta, \gamma\} \ , \ \alpha \neq \beta, \ \beta \neq \gamma, \ \alpha \neq \gamma \qquad (3.12)$$

In other words nonzero coefficients may only occur among those with indexing set consisting of three distinct numbers, and must be zero otherwise.

Let us now turn to ω-part of (3.6). The fact that V_1 preserves ω implies that it is hamiltonian, so its coefficients:

$$b = J\nabla H$$

for some hamiltonian $H = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} h_{ij}x^i x^j$, homogeneous second degree polynomial, i.e.:

$$b^k_s = \pm h_{k\pm n,s} \qquad (3.13)$$

where the sign that makes sense applies:

$$\left[\begin{array}{lll} + & for & k \leq n \\ - & for & k > n \end{array}\right.$$

Our choice of $\Gamma_0 = 0$ implies $V_2 = 0$. Hence second equation in ω-part of (3.6) is satisfied trivially.

(3.12), coupled with compatibility conditions (3.11) leaves us only five types of possibly non-zero coefficients:

$$c^{\alpha m}_{\alpha k},\ m \neq k,\ \{m,k\} \cap \{\alpha,\bar\alpha\} = \emptyset\ , \qquad c^{\alpha k}_{\beta k},\ k \notin \{\alpha,\beta,\bar\alpha,\bar\beta\},\ \alpha \neq \beta\ ,$$

$$c^{\alpha\alpha}_{\bar\alpha k},\ c^{\alpha\bar\alpha}_{\bar\alpha k},\ c^{\alpha k}_{\bar\alpha\bar\alpha},\ k \notin \{\alpha,\bar\alpha\}\ .$$

We will now specify (3.10) to different particular $\{i,j,l,p\}$.
i) $i = \bar P$, $j = l \neq i$

$$(c^{jk}_{ij} - c^{ji}_{kj})b^k_{\bar i} + (c^{j\bar i}_{kj} - c^{jk}_{\bar i j})b^k_i + (c^{ki}_{\bar i j} - c^{k\bar i}_{ij})b^j_k + (c^{j\bar i}_{ik} - c^{ji}_{\bar i k})b^k_j = 0$$

There are 3 distinct indexes present in each coefficient in the above equation. If it seems that there are only 2, we must use their symmetries (3.11), to explicitly present all three. For example, the first coefficient $c^{jk}_{ij} = -c^{\bar i k}_{\bar j j}$, and similar for other coefficients. That implies that in each summation the dummy index k has to turn into one of the fixed ones, e.g. into j, $\bar i$ or $\bar j$ in the first term:

$$(c^{jj}_{ij} - c^{ji}_{jj})b^j_{\bar i} + (c^{j\bar i}_{ij} - c^{ji}_{\bar i j})b^{\bar i}_{\bar i} + (c^{j\bar j}_{ij} - c^{ji}_{\bar j j})b^{\bar j}_{\bar i} = (c^{jj}_{ij} - c^{ji}_{jj})h_{\bar j\bar i} - (c^{j\bar i}_{ij} - c^{ji}_{\bar i j})h_{i\bar i} - (c^{j\bar j}_{ij} - c^{ji}_{\bar j j})h_{j\bar i}$$

Note that even though indexes do repeat in the first term above, the summation convention does not apply, because the designated summation dummy k is absent! After the similar work is done for the remaining 3 terms, and many cancellations (due to symmetries (3.11)),the original equation simplifies to:

$$c^{ji}_{\bar i\bar i}h_{ij} + c^{ii}_{\bar i j}h_{i\bar j} + c^{j\bar i}_{ii}h_{\bar i j} - c^{\bar i\bar i}_{ij}h_{\bar j\bar i} = 0 \qquad i \neq j\ .$$

In the similar manner we obtain the next four equations:
ii) $i = \bar P$, $j = \bar L \neq i$

$$2(c^{j\bar i}_{ji} - c^{ii}_{j\bar j})h_{jj} + (c^{\bar j\bar j}_{j\bar i} - 2c^{ii}_{\bar i j})h_{ij} + c^{\bar j\bar i}_{jj}h_{i\bar j} + (2c^{\bar j\bar i}_{ii} - c^{\bar j\bar j}_{ij})h_{\bar i j} - c^{\bar j i}_{jj}h_{\bar i\bar j} = 0 \qquad i \neq j\ .$$

iii) $\bar I = j(=: i)$, $l = \bar P$, $I < P \Rightarrow j(= i) < l$

$$(c^{l\bar l}_{\bar i i} - c^{l\bar i}_{\bar l i})h_{ii} - c^{l\bar i}_{l\bar l}h_{il} + (2c^{l\bar i}_{ii} - c^{\bar i l}_{\bar l l})h_{\bar i\bar l} + (c^{li}_{\bar l i} - c^{l\bar l}_{ii})h_{\bar i\bar i} + (2c^{l\bar i}_{li} - c^{ll}_{\bar i i} - c^{i\bar l}_{il})h_{\bar l\bar l} = 0 \qquad i < l\ .$$

$\bar I = l$, $j = \bar P$, $I < P \Rightarrow l < j$

$$(c^{l\bar l}_{\bar j j} - c^{l\bar j}_{\bar l j})h_{jj} - c^{l\bar j}_{l\bar l}h_{jl} + (2c^{l\bar j}_{jj} - c^{\bar j l}_{\bar l l})h_{\bar j\bar l} + (c^{lj}_{\bar l j} - c^{l\bar l}_{jj})h_{\bar j\bar j} + (2c^{l\bar j}_{lj} - c^{ll}_{\bar j j} - c^{j\bar l}_{jl})h_{\bar l\bar l} = 0 \qquad j > l\ .$$

Since these two equations are the same modulo changing i into j, we can keep just the last equation, but for $j \neq l$. And finally we rewrite it in i and j in conformity with others:

$$(c^{j\bar j}_{\bar i\bar i} - c^{j\bar i}_{\bar j\bar i})h_{ii} - c^{j\bar i}_{\bar j\bar j}h_{ij} + (2c^{j\bar i}_{ii} - c^{\bar i j}_{\bar j\bar j})h_{\bar i\bar j} + (c^{ji}_{\bar j i} - c^{j\bar j}_{ii})h_{\bar i\bar i} + (2c^{j\bar i}_{ji} - c^{jj}_{\bar i i} - c^{i\bar j}_{ij})h_{\bar j\bar j} = 0 \qquad i \neq j \, .$$

iv) $i = l$, $J(=: \bar j\ ) = P \neq \bar i$

$$(c^{i\bar i}_{\bar j\bar j} - c^{i\bar j}_{\bar i\bar j})h_{ii} + (c^{\bar i i}_{\bar j\bar j} - c^{\bar i\bar j}_{i\bar j})h_{\bar i\bar i} + (c^{ij}_{i\bar j} + c^{i\bar j}_{ij} - 2c^{ii}_{\bar j\bar j})h_{\bar j\bar j} + (2c^{ii}_{\bar i j} - c^{ij}_{\bar j\bar j})h_{i\bar j} + c^{ji}_{\bar j\bar j}h_{\bar i j} = 0 \qquad i \neq j \, .$$

v) $i = \bar L$, $J(=: \bar j\ ) = P \neq \bar i$

$$-c^{ji}_{\bar j\bar j}h_{ij} + 2(c^{\bar i\bar j}_{i\bar j} - c^{\bar i i}_{\bar j\bar j})h_{i\bar i} + (c^{\bar i i}_{\bar j\bar j} - c^{\bar i\bar j}_{i\bar j})h_{j\bar j} + c^{\bar j\bar j}_{ij}h_{\bar j\bar j} + (c^{jj}_{i\bar j} - c^{\bar i\bar i}_{i\bar j} + c^{\bar j i}_{\bar j i} - c^{\bar j\bar j}_{\bar i\bar i})h_{i\bar j} = 0 \qquad i \neq j \, .$$

In each of the equations i)-v) above we are free to interchange i with j to obtain another five: i'), ii'), iii'), iv') and v'). Thus we obtain the system of ten equations for ten variables: h_{ij}, $h_{i\bar j}$, $h_{\bar i j}$, $h_{\bar i\bar j}$, h_{ii}, $h_{\bar i\bar i}$, h_{jj}, $h_{\bar j\bar j}$, $h_{i\bar i}$, and $h_{j\bar j}$. However, the last two variables are only found in the equation v), which allows us to consider first four equations and their 'primes' i), i'),... iv') as an 8 X 8 system for the first eight variables:

$$\begin{array}{c}
\begin{array}{cccccccc}
h_{ij} & h_{i\bar j} & h_{\bar i j} & h_{\bar i\bar j} & h_{ii} & h_{\bar i\bar i} & h_{jj} & h_{\bar j\bar j}
\end{array} \\
\begin{pmatrix}
-c^{ji}_{\bar i\bar i} & -c^{ii}_{\bar i j} & -c^{j\bar i}_{ii} & c^{\bar i\bar i}_{ij} & & & & \\
-c^{ij}_{\bar j\bar j} & -c^{i\bar j}_{jj} & -c^{jj}_{\bar j i} & c^{\bar j\bar j}_{ji} & & & & \\
(c^{\bar j\bar j}_{\bar i j} - 2c^{ii}_{\bar i j}) & c^{\bar j\bar i}_{jj} & (2c^{\bar j\bar i}_{ii} - c^{\bar j\bar j}_{ij}) & -c^{\bar j i}_{jj} & & & 2(c^{j\bar i}_{ji} - c^{ii}_{j\bar j}) & \\
(c^{\bar i\bar i}_{i\bar j} - 2c^{jj}_{\bar j i}) & (2c^{\bar i\bar j}_{jj} - c^{\bar i\bar i}_{ji}) & c^{\bar i\bar j}_{ii} & -c^{\bar i j}_{ii} & 2(c^{i\bar j}_{ij} - c^{jj}_{i\bar i}) & & & \\
-c^{j\bar i}_{\bar j\bar j} & & & (2c^{j\bar i}_{ii} - c^{\bar i j}_{\bar j\bar j}) & (c^{j\bar j}_{\bar i\bar i} - c^{j\bar i}_{\bar j\bar i}) & (c^{ji}_{\bar j i} - c^{j\bar j}_{ii}) & & (2c^{j\bar i}_{ji} - c^{jj}_{\bar i i} - c^{i\bar j}_{ij}) \\
-c^{i\bar j}_{\bar i\bar i} & & & (2c^{i\bar j}_{jj} - c^{\bar j i}_{\bar i\bar i}) & & (2c^{i\bar j}_{ij} - c^{ii}_{\bar j j} - c^{j\bar i}_{ji}) & (c^{i\bar i}_{\bar j\bar j} - c^{i\bar j}_{\bar i\bar j}) & (c^{ij}_{\bar i j} - c^{i\bar i}_{jj}) \\
 & (2c^{ii}_{\bar i j} - c^{ij}_{\bar j\bar j}) & c^{ji}_{\bar j\bar j} & & (c^{i\bar i}_{\bar j\bar j} - c^{i\bar j}_{\bar i\bar j}) & (c^{\bar i i}_{\bar j\bar j} - c^{\bar i\bar j}_{i\bar j}) & & (c^{ij}_{i\bar j} + c^{i\bar j}_{ij} - 2c^{ii}_{\bar j\bar j}) \\
 & c^{ij}_{\bar i\bar i} & (2c^{jj}_{\bar j i} - c^{ji}_{\bar i\bar i}) & & & (c^{ji}_{j\bar i} + c^{j\bar i}_{ji} - 2c^{jj}_{\bar i\bar i}) & (c^{j\bar j}_{\bar i\bar i} - c^{j\bar i}_{\bar j\bar i}) & (c^{\bar j j}_{\bar i\bar i} - c^{\bar j\bar i}_{j\bar i})
\end{pmatrix}
\end{array}$$

We set:

$$2c^{j\bar i}_{ii} - c^{\bar i j}_{\bar j\bar j}(= 2c^{\bar i\bar i}_{i\bar j} - c^{jj}_{\bar j i}) = 0 \, , \quad c^{ii}_{\bar i j}(= c^{ji}_{\bar i\bar i}) = 0 \, , \quad c^{*\alpha}_{**} = 0 \, , * \in \{i, \bar i\} \, , \ \alpha \notin \{i, \bar i\} \, .$$

This doesn't completely separate the system, but it does annihilate the lower left block. Consider the lower right block:

$$
\begin{array}{cccc}
h_{ii} & h_{\bar{i}\bar{i}} & h_{jj} & h_{\bar{j}\bar{j}} \\
\end{array}
$$

$$
\begin{pmatrix}
(c^{i\bar{j}}_{\bar{i}\bar{j}} - c^{j\bar{i}}_{\bar{j}\bar{i}}) & (c^{ji}_{\bar{j}i} + c^{\bar{i}\bar{j}}_{i\bar{j}}) & 0 & (2c^{j\bar{i}}_{ji} - c^{ij}_{i\bar{j}} - c^{i\bar{j}}_{ij}) \\
0 & (2c^{i\bar{j}}_{ij} - c^{ji}_{j\bar{i}} - c^{j\bar{i}}_{ji}) & (c^{j\bar{i}}_{\bar{j}\bar{i}} - c^{i\bar{j}}_{\bar{i}\bar{j}}) & (c^{ij}_{\bar{i}j} + c^{\bar{j}\bar{i}}_{j\bar{i}}) \\
(c^{j\bar{i}}_{\bar{j}\bar{i}} - c^{i\bar{j}}_{\bar{i}\bar{j}}) & -(c^{ji}_{\bar{j}i} + c^{\bar{i}\bar{j}}_{i\bar{j}}) & 0 & (2c^{\bar{j}i}_{\bar{j}\bar{i}} + c^{i\bar{j}}_{ij} + c^{ij}_{i\bar{j}}) \\
0 & (2c^{\bar{i}j}_{\bar{i}\bar{j}} + c^{ji}_{j\bar{i}} + c^{j\bar{i}}_{ji}) & (c^{i\bar{j}}_{\bar{i}\bar{j}} - c^{j\bar{i}}_{\bar{j}\bar{i}}) & -(c^{ij}_{\bar{i}j} + c^{\bar{j}\bar{i}}_{j\bar{i}})
\end{pmatrix}
$$

It has ten independent coefficients, which we can set as follows:

$$
c^{i\bar{j}}_{\bar{i}\bar{j}} = \left[\begin{array}{ll} +1 & i > j \\ -1 & i < j \end{array} \right. \quad , \quad c^{j\bar{i}}_{ji} = c^{i\bar{j}}_{ij} = 2 \text{ , and the rest all equal to 1.}
$$

It is clearly non-degenerate, hence we just need to show that the upper left block also can be chosen non-degenerate:

$$
\begin{array}{cccc}
h_{ij} & h_{i\bar{j}} & h_{\bar{i}j} & h_{\bar{i}\bar{j}} \\
\end{array}
$$

$$
\begin{pmatrix}
0 & -c^{ii}_{\bar{i}j} & -c^{j\bar{i}}_{ii} & c^{\bar{i}\bar{i}}_{ij} \\
0 & -c^{i\bar{j}}_{jj} & -c^{jj}_{\bar{j}i} & c^{\bar{j}\bar{j}}_{ji} \\
-3c^{\bar{j}\bar{j}}_{j\bar{i}} & 0 & (2c^{\bar{j}\bar{i}}_{ii} - c^{\bar{j}\bar{j}}_{ij}) & 0 \\
-3c^{\bar{i}\bar{i}}_{i\bar{j}} & (2c^{\bar{i}\bar{j}}_{jj} - c^{\bar{i}\bar{i}}_{ji}) & 0 & 0
\end{pmatrix}
=
\begin{pmatrix}
0 & -c^{ii}_{\bar{i}j} & \frac{1}{2}c^{\bar{j}i}_{jj} & c^{\bar{i}\bar{i}}_{ij} \\
0 & \frac{1}{2}c^{i\bar{j}}_{jj} & -c^{jj}_{\bar{j}i} & c^{\bar{j}\bar{j}}_{ji} \\
-\frac{3}{2}c^{ii}_{\bar{i}j} & 0 & (2c^{\bar{i}\bar{i}}_{ij} - c^{\bar{j}\bar{j}}_{ji}) & 0 \\
-\frac{3}{2}c^{jj}_{\bar{j}i} & (2c^{\bar{j}\bar{j}}_{ji} - c^{\bar{i}\bar{i}}_{ij}) & 0 & 0
\end{pmatrix}
$$

There are four independent coefficients in this system:

$$
c^{ii}_{\bar{i}j}\ ,\ c^{\bar{j}i}_{jj}\ ,\ c^{\bar{i}\bar{i}}_{ij} \ \text{ and } \ c^{\bar{j}\bar{j}}_{ji}\ .
$$

Setting them all equal to 1 achieves non-degeneracy for this block, and for the 8 X 8 system. This leaves us equations v) and v'), which reduce to this 2 X 2 system for the remaining two unknowns $h_{i\bar{i}}$ and $h_{j\bar{j}}$:

$$
\begin{pmatrix}
2(c^{\bar{i}\bar{j}}_{i\bar{j}} + c^{ji}_{\bar{j}i}) & -(c^{ji}_{\bar{j}i} + c^{\bar{i}\bar{j}}_{i\bar{j}}) \\
-(c^{ij}_{\bar{i}j} + c^{\bar{j}\bar{i}}_{j\bar{i}}) & 2(c^{\bar{j}\bar{i}}_{j\bar{i}} + c^{ij}_{\bar{i}j})
\end{pmatrix}
=
\begin{pmatrix}
4 & -2 \\
-2 & 4
\end{pmatrix}
$$

Since we required 3 distinct indexes for our non-zero coefficients, this method is only good for dimensions 4 or larger.

Proposition 3.2 *The stabilizer of a k-jet of a generic connection for $n \geq 4$ is: G_1/G_2 for $k = 0$, and 0 for $k \geq 1$.*

The lowest dimension 2 has to be treated separately.

4 Exceptional dimension 2

In this case the stabilizer of the 1-jet is non-trivial (it has dimension one), stabilizers of the higher jets are all trivial.
Since the general method of previous section fails here, we must reconsider (3.10) with $i = 1$ and $p = 2$:

$$(c^{lk}_{1j} - c^{l1}_{kj})b^k_2 + (c^{l2}_{kj} - c^{lk}_{2j})b^k_1 + (c^{k1}_{2j} - c^{k2}_{1j})b^l_k + (c^{l2}_{1k} - c^{l1}_{2k})b^k_j = 0$$

Summing over two indexes, we obtain:

$$(c^{11}_{2j} - c^{12}_{1j})b^l_1 + (c^{21}_{2j} - c^{22}_{1j})b^l_2 + (c^{l2}_{11} - c^{l1}_{21})h_{2j} - (c^{l2}_{12} - c^{l1}_{22})h_{1j} = 0$$

Varying pair (ij) we obtain next 4 equations on 3 variables h_{11}, h_{12} and h_{22}:

$$(c^{11}_{21} - c^{12}_{11})h_{21} + (c^{21}_{21} - c^{22}_{11})h_{22} + (c^{12}_{11} - c^{11}_{21})h_{21} - (c^{12}_{12} - c^{11}_{22})h_{11} = 0$$

$$-(c^{11}_{21} - c^{12}_{11})h_{11} - (c^{21}_{21} - c^{22}_{11})h_{12} + (c^{22}_{11} - c^{21}_{21})h_{21} - (c^{22}_{12} - c^{21}_{22})h_{11} = 0$$

$$(c^{11}_{22} - c^{12}_{12})h_{21} + (c^{21}_{22} - c^{22}_{12})h_{22} + (c^{12}_{11} - c^{11}_{21})h_{22} - (c^{12}_{12} - c^{11}_{22})h_{12} = 0$$

$$-(c^{11}_{22} - c^{12}_{12})h_{11} - (c^{21}_{22} - c^{22}_{12})h_{12} + (c^{22}_{11} - c^{21}_{21})h_{22} - (c^{22}_{12} - c^{21}_{22})h_{12} = 0$$

The coefficient matrix of the system is this:

$$\begin{array}{ccc} h_{11} & h_{12} & h_{22} \\ \left(\begin{array}{c} -(c^{12}_{12} - c^{11}_{22}) \\ \\ -(c^{11}_{21} - c^{12}_{11} + c^{22}_{12} - c^{21}_{22}) \\ \\ \\ \\ -(c^{11}_{22} - c^{12}_{12}) \end{array}\right. & \begin{array}{c} \\ \\ -2(c^{21}_{21} - c^{22}_{11}) \\ \\ 2(c^{11}_{22} - c^{12}_{12}) \\ \\ \\ \end{array} & \left.\begin{array}{c} (c^{21}_{21} - c^{22}_{11}) \\ \\ \\ \\ (c^{21}_{22} - c^{22}_{12} + c^{12}_{11} - c^{11}_{21}) \\ \\ (c^{22}_{11} - c^{21}_{21}) \end{array}\right) \end{array}$$

Setting

$$a := c^{12}_{11} - c^{11}_{21} \ , \quad b := c^{12}_{12} - c^{11}_{22} \ , \quad \text{and } c := c^{21}_{21} - c^{22}_{11} \ ,$$

we transform it into:

$$\begin{array}{c} \begin{array}{ccc} h_{12} & h_{11} & h_{22} \end{array} \\ \begin{pmatrix} 0 & -b & c \\ 0 & b & -c \\ -2c & 2a & 0 \\ -2b & 0 & 2a \end{pmatrix} \end{array}$$

It is clearly degenerate, and has rank 2 in general position.

This means we need to consider (3.8) in full generality: for arbitrary Γ_0 and Γ_1. ((3.10) is (3.8) under assumption that $\Gamma_0 = 0$, which now has to be lifted.) (3.8) involves $\widetilde{\mathcal{L}}_{V_2}\Gamma_0$, so we need to express V_2 from the first equation of Γ-part of (3.6):

$$\mathcal{L}_{V_1}\Gamma_0 + \frac{\partial^2 V_2}{\partial x^2} = 0\,.$$

Setting $\Gamma^k_{0ij} =: \gamma^k_{ij}$, it can be rewritten in index form as:

$$V^l_{2,ij} = \gamma^k_{ij} b^l_k - \gamma^l_{kj} b^k_i - \gamma^l_{ik} b^k_j =: v^l_{ij}$$

Actually, second derivatives of V_2 is all we need in (3.8), where they appear in

$$(\widetilde{\mathcal{L}}_{V_2}\Gamma_0)^l_{ij,p} = -\gamma^k_{ij} V^l_{2,kp} + \gamma^l_{kj} V^k_{2,ip} + \gamma^l_{ik} V^k_{2,jp}\ ,$$

which we can now rewrite as:

$$\begin{aligned}(\widetilde{\mathcal{L}}_{V_2}\Gamma_0)^l_{ij,p} = -\gamma^k_{ij}(\gamma^s_{kp} b^l_s - \gamma^l_{sp} b^s_k - \gamma^l_{ks} b^s_p) + \gamma^l_{kj}(\gamma^s_{ip} b^k_s - \gamma^k_{sp} b^s_i - \gamma^k_{is} b^s_p)\\ + \gamma^l_{ik}(\gamma^s_{jp} b^k_s - \gamma^k_{sp} b^s_j - \gamma^k_{js} b^s_p)\end{aligned} \tag{4.14}$$

One note about coefficients γ. Compatibility conditions (1.1) in dimension $n = 2$ become:

$$\gamma^1_{1j} = -\gamma^2_{2j}\ .$$

That leaves 4 independent coefficients: γ^2_{11}, γ^1_{11}, γ^1_{12} and γ^1_{22}.

We consider (3.8) as $S(V_1) = 0$ - linear operator acting on V_1, and split the operator into two parts: $S = S(\Gamma_0) + S(\Gamma_1)$. Matrix of $S(\Gamma_1)$ is calculated at the top of this section.

(4.14) allows us to rewrite $S(\Gamma_0)V_1 = (\widetilde{\mathcal{L}}_{V_2}\Gamma_0)^l_{ij,p} - (\widetilde{\mathcal{L}}_{V_2}\Gamma_0)^l_{pj,i}$ as:

$$\begin{aligned}(\gamma^k_{pj}\gamma^s_{ki} - \gamma^k_{ij}\gamma^s_{kp}) b^l_s + (\gamma^l_{pk}\gamma^k_{js} - \gamma^k_{pj}\gamma^l_{ks}) b^s_i +\\ + (\gamma^l_{pk}\gamma^k_{si} - \gamma^l_{ik}\gamma^k_{sp}) b^s_j + (\gamma^k_{ij}\gamma^l_{ks} - \gamma^l_{ik}\gamma^k_{js}) b^s_p\end{aligned} \tag{4.15}$$

Recall that dimension $n = 2$, and indexes $1 = i < p = 2$ must therefore stay fixed at $i = 1$, $p = 2$, while the remaining pair of indexes take any values. That turns (4.15) into a system of 4 expressions indexed with (j, l):

$$\begin{aligned}(11)\qquad & (\gamma^1_{22}\gamma^2_{11} - \gamma^1_{12}\gamma^2_{12}) b^1_1 + (\gamma^2_{11}\gamma^1_{22} - \gamma^1_{12}\gamma^2_{12}) b^2_2 +\\ & + (\gamma^1_{2k}\gamma^k_{12} - \gamma^1_{1k}\gamma^k_{22}) b^2_1 + (\gamma^k_{21}\gamma^2_{k1} - \gamma^k_{11}\gamma^2_{k2}) b^1_2\end{aligned}$$

$$\begin{aligned}(22)\qquad & (\gamma^2_{21}\gamma^1_{21} - \gamma^1_{22}\gamma^2_{11}) b^1_1 + (\gamma^1_{12}\gamma^2_{12} - \gamma^2_{11}\gamma^1_{22}) b^2_2 +\\ & + (\gamma^k_{22}\gamma^1_{k1} - \gamma^k_{12}\gamma^1_{k2}) b^2_1 + (\gamma^2_{2k}\gamma^k_{11} - \gamma^2_{1k}\gamma^k_{21}) b^1_2\end{aligned}$$

(12) $\quad 2(\gamma^2_{2k}\gamma^k_{11} - \gamma^k_{21}\gamma^2_{k1})b^1_1 + \qquad 2(\gamma^2_{21}\gamma^1_{21} - \gamma^2_{11}\gamma^1_{22})b^2_1$

(21) $\quad 2(\gamma^1_{2k}\gamma^k_{21} - \gamma^1_{1k}\gamma^k_{22})b^2_2 + \qquad 2(\gamma^1_{22}\gamma^2_{11} - \gamma^1_{12}\gamma^2_{12})b^1_2$

We use (3.13, V_1-hamiltonian) to go from b-coefficients for V_1 to h-coefficients. Then the fact that V_2 too is hamiltonian (second equation in ω-part of (3.6)) follows automatically, as a short calculation would show. Considered by itself, this system is degenerate. Indeed, setting

$$(\gamma^1_{22}\gamma^2_{11} - \gamma^1_{12}\gamma^2_{12}) =: A, (\gamma^1_{2k}\gamma^k_{12} - \gamma^1_{1k}\gamma^k_{22}) =: B, (\gamma^k_{21}\gamma^2_{k1} - \gamma^k_{11}\gamma^2_{k2}) =: C \ ,$$

and using h-coefficients, the system's matrix becomes:

$$\begin{array}{c} \begin{array}{ccc} h_{12} & h_{11} & h_{22} \end{array} \\ \begin{pmatrix} 0 & -B & C \\ 0 & B & -C \\ -2C & 2A & 0 \\ -2B & 0 & 2A \end{pmatrix} \end{array}$$

The determinant of this is identically zero.
Notice that the two matrices for $S(\Gamma_0)$ and $S(\Gamma_1)$ obtained so far look exactly the same, up to capitalization of the entries' names. The matrix for the operator $S = S(\Gamma_0) + S(\Gamma_1)$ is a sum of the two. Since it will have the same structure as either of its degenerate summands, it is also degenerate. It has rank 2 however, since it's lower right 2 X 2 block is:

$$\begin{pmatrix} 2(a+A) & 0 \\ 0 & 2(a+A) \end{pmatrix}$$

This is non-degenerate in general position, since:

$$a + A = \gamma^1_{22}\gamma^2_{11} + \gamma^1_{12}\gamma^1_{11} - (c^{22}_{21} + c^{11}_{12}) \neq 0 \ ,$$

resulting in a 1-dimensional stabilizer at 1-jet.

Let us now consider the next, second jet of our connection. To calculate its stabilizer, we need to solve the following equation from (3.6) for V_4 :

$$\mathcal{L}_{V_1}\Gamma_2 + \tilde{\mathcal{L}}_{V_2}\Gamma_1 + \tilde{\mathcal{L}}_{V_3}\Gamma_0 + \frac{\partial^2 V_4}{\partial x^2} = 0$$

Its compatibility conditions are:

$$(\mathcal{L}_{V_1}\Gamma_2 + \tilde{\mathcal{L}}_{V_2}\Gamma_1 + \tilde{\mathcal{L}}_{V_3}\Gamma_0)^l_{ij,p} = (\mathcal{L}_{V_1}\Gamma_2 + \tilde{\mathcal{L}}_{V_2}\Gamma_1 + \tilde{\mathcal{L}}_{V_3}\Gamma_0)^l_{pj,i} \qquad (4.16)$$

We will use the same strategy as in the previous section to prove that in this case stabilizer is trivial. Namely we will obtain a connection 2-jet, for which the above equation will be a non-degenerate homogeneous linear system. We

set $\Gamma_0 = \Gamma_1 = 0$. This implies $V_2 = V_3 = 0$, hence hamiltonian, so that ω-part of (3.6) is true for any hamiltonian V_1. That simplifies (4.16) to:

$$(\mathcal{L}_{V_1}\Gamma_2)^l_{ij,p} = (\mathcal{L}_{V_1}\Gamma_2)^l_{pj,i} \tag{4.17}$$

We introduce notation for coefficients of Γ_2 :

$$\Gamma_2{}^l_{ij} = \sum_{s,t=1}^{2} d^l_{ijst} x^s x^t \ , \ d^l_{ijst} = d^l_{jits}$$

Compatibility with ω (1.1) impose these restrictions on d in dimension 2:

$$d^2_{2\alpha ij} = -d^1_{1\alpha ij}$$

There are thus 4 families of independent coefficients: d^2_{11ij}, d^1_{11ij}, d^1_{12ij} and d^1_{22ij}. With these,

$$(\mathcal{L}_{V_1}\Gamma_2)^l_{ij,p} = 2d^l_{ijkt}b^k_p x^t + 2d^l_{ijkp}b^k_t x^t - 2d^k_{ijpt}b^l_k x^t + 2d^l_{kjpt}b^k_i x^t + 2d^l_{ikpt}b^k_j x^t$$

(b^l_k are still coefficients of V_1, as in section (5), and (4.17) (with $i = 1, p = 2$) is:

$$(d^l_{kj2t} - d^l_{2jkt})b^k_1 + (d^l_{1jkt} - d^l_{kj1t})b^k_2 + (d^l_{1jk2} - d^l_{2jk1})b^k_t +$$
$$(d^k_{2j1t} - d^k_{1j2t})b^l_k + (d^l_{1k2t} - d^l_{2k1t})b^k_j = 0$$

With the triple of indexes (j, l, t) arbitrary, we have system of 8 equations in 4 variables: the coefficients of V_1. This is the system, equations are labelled by this index triple:

(111) $$2(d^1_{1112} - d^1_{1211})b^1_1 + (d^1_{1112} - d^1_{1211})b^2_2 + (d^1_{1112} - d^1_{1211})b^2_1 + (d^2_{1211} - d^2_{1112})b^1_2 = 0$$

(221) $$2(d^2_{1212} - d^2_{2211})b^1_1 + (d^2_{1212} - d^2_{2211})b^2_2 + (d^2_{1222} - d^2_{2221} + d^1_{2211} - d^1_{1212})b^2_1 + (d^2_{1112} - d^2_{1211})b^1_2 = 0$$

(121) $$3(d^2_{1112} - d^2_{1211})b^1_1 + (d^2_{1122} - d^2_{2211} + d^1_{2111} - d^1_{1112})b^2_1 = 0$$

(211) $$(d^1_{1212} - d^1_{2211})b^1_1 + 2(d^1_{1212} - d^1_{2211})b^2_2 + (d^1_{1222} - d^1_{2221})b^2_1 + (d^1_{1112} - d^1_{2111} + d^2_{2211} - d^2_{1221})b^1_2 = 0$$

(112) $$(d^1_{1122} - d^1_{1212})b^1_1 + 2(d^1_{1122} - d^1_{1212})b^2_2 + (d^1_{1222} - d^1_{2221})b^2_1 + (d^1_{1112} - d^1_{2111} + d^2_{1212} - d^2_{1122})b^1_2 = 0$$

(222) $$(d^2_{1222} - d^2_{2221})b^1_1 + 2(d^2_{1222} - d^2_{2221})b^2_2 + (d^1_{2221} - d^1_{1222})b^2_1 + (d^2_{1122} - d^2_{2211})b^1_2 = 0$$

$$
\begin{aligned}
(122)\qquad & 2(d^2_{1122}-d^2_{1212})b^1_1+(d^2_{1122}-d^2_{1212})b^2_2+ \\
& (d^2_{1222}-d^2_{2221}+d^1_{1212}-d^1_{1122})b^2_1+(d^2_{1112}-d^2_{2111})b^1_2=0
\end{aligned}
$$

$$
(212)\qquad 3(d^1_{1222}-d^1_{2212})b^2_2+\qquad (d^2_{2221}-d^2_{1222}+d^1_{1122}-d^1_{2211})b^1_2=0
$$

Setting:

$a=(d^1_{1112}-d^1_{1211}), e=(d^1_{1212}-d^1_{1122}), g=(d^1_{1222}-d^1_{2221}), h=(d^1_{1122}-d^1_{1212})$,

$b=(d^2_{1211}-d^2_{1112}), c=(d^2_{1212}-d^2_{2211}), d=(d^2_{1222}-d^2_{2221}), f=(d^2_{1122}-d^2_{1212})$,

we see the system take form:

$$
\begin{array}{cc}
\begin{array}{cccc} b^1_1 & b^2_2 & b^2_1 & b^1_2 \end{array} & \begin{array}{ccc} h_{12} & h_{11} & h_{22} \end{array} \\
\begin{pmatrix}
2a & a & a & b \\
2c & c & d+e & -b \\
-3b & 0 & f+c-a & 0 \\
-e & -2e & g & a-c \\
h & 2h & g & a-f \\
d & 2d & -g & f+c \\
2f & f & d-h & -b \\
0 & 3g & 0 & h-e-d
\end{pmatrix}
=
&
\begin{pmatrix}
a & -a & b \\
c & -d-e & -b \\
-3b & a-f-c & 0 \\
e & -g & a-c \\
-h & -g & a-f \\
-d & g & f+c \\
f & h-d & -b \\
-3g & 0 & h-e-d
\end{pmatrix}
\end{array}
$$

This is non-degenerate for a generic connection. For example, if $d^2_{1112}=d^2_{1122}=1$, the rest is null, then $f=1$, $b=-1$, all others zero, and the system is:

$$
\begin{pmatrix}
 & & & -1 \\
 & & & 1 \\
3 & -1 & & \\
 & & & \\
 & & & -1 \\
 & & & 1 \\
1 & & & 1 \\
 & & &
\end{pmatrix}
$$

Now we can summarize what we know about exceptional stabilizers:

Proposition 4.1 *The stabilizer of a k-jet of a generic connections for $n=2$ is equal to G_1/G_2 for $k=0$, is 1-dimensional for $k=1$, and is trivial for $k\geq 2$.*

5 Poincaré series

Here we will calculate the Poincaré series of $\mathcal{M}$, the moduli space of Fedosov structures:

$$
p_\Phi(t)=\dim\mathcal{M}_0+\sum_{k=1}^{\infty}(\dim\mathcal{M}_k-\dim\mathcal{M}_{k-1})t^k
$$

To obtain $\dim \mathcal{M}_k$, we need to discuss $\mathcal{F}_k$ first. In particular, we need to know how many different local symplectic structures are there. More precisely, we want to find the dimension of the space of k-jets of non-degenerate closed 2-forms at a point. Non-degeneracy is an open condition and does not affect dimension. Closedness locally is equivalent to exactness. For a symplectic form ω:

$$\omega = d\alpha \ ,$$

for some 1-form α defined up to ∇f, a gradient of a function, that function in its turn is defined up to a constant. We have the following exact sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(\mathbb{R}^{2n}) \xrightarrow{d^0} \Lambda^1(\mathbb{R}^{2n}) \xrightarrow{d^1} d\Lambda^1(\mathbb{R}^{2n}) \longrightarrow 0 \ ,$$

which descends to jets:

$$0 \longrightarrow \mathbb{R} \longrightarrow j^{l+2}(C^\infty(\mathbb{R}^{2n})) \xrightarrow{d^0} j^{l+1}(\Lambda^1(\mathbb{R}^{2n})) \xrightarrow{d^1} j^l(d\Lambda^1(\mathbb{R}^{2n})) \xrightarrow{d^2} 0$$

It follows that:

$$\dim[j^l d\Lambda^1(\mathbb{R}^{2n})] = \dim[j^{l+1}\Lambda^1(\mathbb{R}^{2n})] - \dim[j^{l+2}(C^\infty(\mathbb{R}^{2n}))] + \dim \mathbb{R}$$

We are interested in 0-jets since higher jets of ω are determined by the connection part Γ of a given Fedosov structure Φ through compatibility condition (1.1), see Theorem 4.5 (2) p.124 in [GRS].

$$\dim[j^0 d\Lambda^1(\mathbb{R}^{2n})] = \dim[j^1\Lambda^1(\mathbb{R}^{2n})] - \dim[j^2(C^\infty(\mathbb{R}^{2n}))] + 1 =$$

$$= 2n\binom{2n+1}{2n} - \binom{2n+2}{2n} + 1 = \frac{2n(2n-1)}{2}$$

Each ω is compatible with (or preserved by) all Γ, such that $\omega_{i\alpha}\Gamma^\alpha_{jk}$ is completely symmetric in i, j, k, cf. the last paragraph on p.110 in [GRS]. At 0-jet of Fedosov structure $\Phi_0 = (\omega_0, \Gamma_0)$ there are $\binom{2n+3-1}{2n-1}$ of those, hence:

$$\dim \mathcal{F}_0 = \dim\{\text{all } \omega_0\} \dim\{\text{all compatible } \Gamma_0\} = \frac{2n(2n-1)}{2}\binom{2n+2}{2n-1}$$

For other $\mathcal{F}_k$'s we must remember that each Γ^i_{jk} is a homogeneous polynomial of degree k in $2n$ variables:

$$\dim \mathcal{F}_k = \frac{2n(2n-1)}{2}\binom{2n+2}{2n-1}\sum_{m=0}^{k}\binom{2n+m-1}{2n-1} = \frac{2n(2n-1)}{2}\binom{2n+2}{2n-1}\binom{2n+k}{2n}$$

Next, we need to know orbit dimensions. G_1/G_3 acts on Φ_0 non-trivially, i.e. both first and second component of generating vector field V_1 and V_2 are acting. The stabilizer G_{Φ_0} is determined by an arbitrary hamiltonian V_1:

$$\dim \mathcal{O}_0 = \dim\{(V_1, V_2)\} - \dim \mathtt{sp}(2n)$$

$$= 2n\sum_{m=1}^{2}\binom{2n+m-1}{2n-1} - \frac{2n(2n+1)}{2} = n((2n+1)^2-2) = n(4n^2+4n-1)$$

V_1,..., V_{k+2} act on Φ_k:

$$\dim \mathcal{O}_k = \dim\{(V_1,\ldots,V_{k+2})\} - 1\cdot\delta^2_{2n}\delta^1_k$$

(Kronecker symbol δ is needed here to take care of exceptional dimension two.)

$$= 2n\sum_{m=1}^{k+2}\binom{2n+m-1}{2n-1} - \delta^2_{2n}\delta^1_k = 2n\left[\binom{2n+k+2}{2n}-1\right] - \delta^2_{2n}\delta^1_k$$

This gives us dimension of moduli space of k-jets:

$$\dim \mathcal{M}_0 = \frac{2n(2n-1)}{2}\binom{2n+2}{2n-1} - n((2n+1)^2-2) = \frac{n[8n(2n^2-1)(n+1)+11]}{6}$$

$$\dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k$$

$$= \frac{2n(2n-1)}{2}\binom{2n+2}{2n-1}\sum_{m=0}^{k}\binom{2n+m-1}{2n-1} - 2n\sum_{m=1}^{k+2}\binom{2n+m-1}{2n-1} + \delta^2_{2n}\delta^1_k$$

$$= \frac{2n(2n-1)}{2}\binom{2n+2}{2n-1}\binom{2n+k}{2n} - 2n\left[\binom{2n+k+2}{2n}-1\right] + \delta^2_{2n}\delta^1_k \text{ , } k\geq 1$$

We will have to write constant and linear terms of Poincaré series separately because they contain $\mathcal{M}_0$. The linear coefficient is:

$$\dim \mathcal{M}_1 - \dim \mathcal{M}_0 = \frac{n(2n+1)}{3}\left[4n^4+2n^3-6n^2-4n-3\right] + \delta^2_{2n}$$

The common term will have this coefficient:

$$\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1} = \frac{2n(2n-1)}{2}\binom{2n+2}{2n-1}\binom{2n+k-1}{2n-1} - 2n\binom{2n+(k+2)-1}{2n-1}$$

$$= 4n\binom{2n+2}{2n-2}\binom{2n+k-1}{2n-1} - 2n\binom{2n+k+1}{2n-1} - \delta^2_{2n}\delta^2_k \text{ , } k\geq 2$$

We have:

$$p_\Phi(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty}(\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1})t^k$$

$$= \frac{n[8n(2n^2-1)(n+1)+11]}{6} + \frac{n(2n+1)}{3}\left[4n^4+2n^3-6n^2-4n-3\right]t + (t-t^2)\delta^2_{2n} +$$

$$+2n\sum_{k=2}^{\infty}\left[2\binom{2n+2}{4}\binom{2n+k-1}{2n-1} - \binom{2n+k+1}{2n-1}\right]t^k$$

Proposition 5.1 *The Poncaré series $p_\Phi(t)$ is a rational function. Namely,*

$$p_\Phi(t) = \frac{n(20n^2+8n+11)}{6} - \frac{n(2n+1)}{3}\left[4n^4+2n^3+2n^2-4n+3\right]t$$
$$+(t-t^2)\delta_{2n}^2 + 2nD_\Phi\left(\frac{1}{1-t}\right)$$

where D_Φ is a differential operator of order $2n-1$:

$$D_\Phi = 2\binom{2n+2}{4}\binom{2n+t\frac{d}{dt}-1}{2n-1} - \binom{2n+t\frac{d}{dt}+1}{2n-1}$$

with

$$\binom{2n+t\frac{d}{dt}-1}{2n-1} = \frac{1}{(2n-1)!}(t\frac{d}{dt}+1)\dots(t\frac{d}{dt}+2n-1)\ ,$$
$$\binom{2n+t\frac{d}{dt}+1}{2n-1} = \frac{1}{(2n-1)!}(t\frac{d}{dt}+3)\dots(t\frac{d}{dt}+2n+1)\ .$$

Proof Indeed, denote

$$\varphi_m(t) = \sum_{k=0}^{\infty} k^m t^k\ , \qquad m \in \mathbb{Z}_+\ ,$$

then

$$\varphi_m(t) = \sum_{k=0}^{\infty} k^{m-1}kt^{k-1}t = t\left(\sum_{k=0}^{\infty} k^{m-1}t^k\right)' = \left(t\frac{d}{dt}\right)\varphi_{m-1}(t) \quad \text{for } m \in \mathbb{N}\,.$$

Thus

$$\varphi_m(t) = \left(t\frac{d}{dt}\right)^m \varphi_0(t) = \left(t\frac{d}{dt}\right)^m\left(\frac{1}{1-t}\right)\ .$$

Hence,

$$\sum_{k=0}^{\infty}\left[2\binom{2n+2}{4}\binom{2n+k-1}{2n-1} - \binom{2n+k+1}{2n-1}\right]t^k$$
$$= \left[2\binom{2n+2}{4}\binom{2n+t\frac{d}{dt}-1}{2n-1} - \binom{2n+t\frac{d}{dt}+1}{2n-1}\right]\left(\frac{1}{1-t}\right)\ .$$

We have:

$$\sum_{k=2}^{\infty}\left[2\binom{2n+2}{4}\binom{2n+k-1}{2n-1} - \binom{2n+k+1}{2n-1}\right]t^k$$
$$= \sum_{k=0}^{\infty}\left[2\binom{2n+2}{4}\binom{2n+k-1}{2n-1} - \binom{2n+k+1}{2n-1}\right]t^k$$
$$-\binom{2n+2}{3}(2n^2-n-1)t - \binom{2n+1}{2}\frac{2n^2+n-4}{3}$$

So

$$
\begin{aligned}
p_\Phi(t) = {} & \frac{n[8n(2n^2-1)(n+1)+11]}{6} + \frac{n(2n+1)}{3}\left[4n^4+2n^3-6n^2-4n-3\right]t \\
& +(t-t^2)\delta_{2n}^2 + 2n\left\{\sum_{k=0}^{\infty}\left[2\binom{2n+2}{4}\binom{2n+k-1}{2n-1} - \binom{2n+k+1}{2n-1}\right]t^k \right. \\
& \left. -\binom{2n+2}{3}(2n^2-n-1)t - \binom{2n+1}{2}\frac{2n^2+n-4}{3}\right\} \\
= {} & \frac{n(20n^2+8n+11)}{6} - \frac{n(2n+1)}{3}\left[4n^4+2n^3+2n^2-4n+3\right]t \\
& +(t-t^2)\delta_{2n}^2 + 2nD_\Phi\left(\frac{1}{1-t}\right)
\end{aligned}
$$

□

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